HOSSAM GHANEM

(9) 7.5 Derivatives Of Inverse Trigonometric Functions (B)

Limits	
$\lim_{x\to -1^+} \sin^{-1} x = -\frac{\pi}{2}$	$\lim_{x\to 1^-} \sin^{-1} x = \frac{\pi}{2}$
$\lim_{x\to -1^+} \cos^{-1} x = \pi$	$\lim_{x\to 1^-} \cos^{-1} x = 0$
$\lim_{x\to \infty} \tan^{-1} x = \frac{\pi}{2}$	$\lim_{x\to -\infty} \tan^{-1} x = -\frac{\pi}{2}$
$\lim_{x\to \infty} \cot^{-1} x = 0$	$\lim_{x\to -\infty} \cot^{-1} x = \pi$

Differentiation
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

Example 1 Show that $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$

Solution

Let $y = \cos^{-1} x$

$$\cos y = x$$
$$-\sin y \cdot y' = 1$$
$$y' = \frac{-1}{\sin y}$$
$$\frac{dy}{dx} = \frac{-1}{\sqrt{\sin^2 y}}$$
$$\frac{d}{dx}(y) = \frac{-1}{\sqrt{1-\cos^2 y}}$$
$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

Example 2 Show that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

Solution

Let $y = \tan^{-1} x$

$$\tan y = x$$
$$\sec^2 y \; y' = 1$$
$$y' = \frac{1}{\sec^2 y}$$
$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Example 3 Show that $\frac{d}{dx}\csc^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$

8 October 1997
14 March 2002

Solution

Let $y = \csc^{-1} x$

$$\csc y = x$$
$$-\csc y \cot y \;\; y' = 1$$
$$y' = \frac{-1}{\csc y \cot y}$$
$$\frac{dy}{dx} = \frac{-1}{\csc y \sqrt{\cot^2 y}}$$
$$\frac{d}{dx}(y) = \frac{-1}{\csc y \sqrt{\csc^2 y - 1}}$$
$$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

Example 4 Show that $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$, $-1 < x < 1.$ 24 March 2008 A

Solution

Let $y = \sin^{-1} x$

$\sin y = x$

$\cos y \cdot y' = 1$

$y' = \dfrac{1}{\cos y}$

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{\cos^2 x}}$

$\dfrac{d}{dx}(y) = \dfrac{1}{\sqrt{1-\sin^2 x}}$

$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

Example 5 Show that $\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1+x^2}$

Solution

Let $y = \cot^{-1} x$

$\cot y = x$

$-\csc^2 y \; y' = 1$

$y' = \dfrac{-1}{\csc^2 y}$

$\dfrac{dy}{dx} = \dfrac{-1}{1+\cot^2 y}$

$\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1+x^2}$

Example 6 Show that $\dfrac{d}{dx} \sec^{-1} x = \dfrac{1}{x\ \sqrt{x^2-1}}$ 8 October 1997 14 March 2002

Solution

Let $y = \sec^{-1} x$

$\sec y = x$

$\sec y \ \tan y \ \ y' = 1$

$y' = \dfrac{1}{\operatorname{Sec} y \tan y}$

$\dfrac{dy}{dx} = \dfrac{1}{\sec y \sqrt{\tan^2 x}}$

$\dfrac{d}{dx}(y) = \dfrac{1}{\sec y \sqrt{\sec^2 y - 1}}$

$\dfrac{d}{dx} \sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}}$

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Example 7

Show that the function defined by 6 March 1997

$$f(x) = \arcsin x - 2\arctan\sqrt{\frac{1+x}{1-x}} \quad , \quad (0 \le x < 1)$$

is a constant function.

Solution

$$f(x) = \sin^{-1} x - 2\tan^{-1}\sqrt{\frac{1+x}{1-x}}$$

$$f^{\backslash}(x) = \frac{1}{\sqrt{1-x^2}} - 2\cdot\frac{1}{\frac{1+x}{1-x}+1}\cdot\frac{1}{2\sqrt{\frac{1+x}{1-x}}}\cdot\frac{1-x-(1+x)}{(1-x)^2}$$

$$= \frac{1}{\sqrt{1-x^2}} - 2\frac{1-x}{1+x+1-x}\cdot\frac{\sqrt{1-x}}{2\sqrt{1+x}}\cdot\frac{2}{(1-x)^2} = \frac{1}{\sqrt{1-x^2}} - 2\frac{(1-x)}{2}\cdot\frac{\sqrt{1-x}}{2\sqrt{1+x}}\cdot\frac{2}{(1-x)^2}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{\sqrt{1-x}}{\sqrt{1+x}}\cdot\frac{1}{1-x} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1+x}}\cdot\frac{1}{\sqrt{1-x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

$\therefore f$ is a constant function.

Example 8

15 July 2003 A

Prove the following identity: $2\sin^{-1} x = \cos^{-1}(1-2x^2)$, $x \ge 0$

Solution

$f(x) = 2\sin^{-1} x \qquad f^{\backslash}(x) = \dfrac{2}{\sqrt{1-x^2}}$

$g(x) = \cos^{-1}(1-2x^2)$

$$g^{\backslash}(x) = \frac{-(-4x)}{\sqrt{1-(1-2x^2)^2}} = \frac{4x}{\sqrt{1-(1-4x^2+4x^4)}} = \frac{4x}{\sqrt{4x^2-4x^4}} = \frac{4x}{\sqrt{4x^2(1-x^2)}}$$

$$= \frac{4x}{2|x|\sqrt{1-x^2}} = \frac{4x}{2x\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

$\therefore f^{\backslash}(x) = g^{\backslash}(x)$

$f(x) = g(x) + C$

$2\sin^{-1} x = \cos^{-1}(1-2x^2) + C$

at $x = 0$

$2\sin^{-1} 0 = \cos^{-1}(1) + C$

$0 = 0 + C \qquad C = 0$

$\therefore 2\sin^{-1} x = \cos^{-1}(1-2x^2)$

Example 9

23 Nove.2007 A

Show that: $\sec^{-1}\left(\frac{e^x}{2}\right) = \sin^{-1}(-2e^{-x}) + \frac{\pi}{2}$, for $x \geq \ln 2$

Solution

$f(x) = \sec^{-1}\left(\frac{e^x}{2}\right)$ $\qquad f^\backslash(x) = \dfrac{\frac{e^x}{2}}{\frac{e^x}{2}\sqrt{\left(\frac{e^x}{2}\right)^2 - 1}} = \dfrac{1}{\sqrt{\left(\frac{e^x}{2}\right)^2 - 1}} = \dfrac{2}{\sqrt{e^{2x} - 4}}$

$g(x) = \sin^{-1}(-2e^{-x}),$ $\qquad g^\backslash(x) = \dfrac{-2(-e^{-x})}{\sqrt{1-(-2e^{-x})^2}} = \dfrac{2e^{-x}}{\sqrt{1-4e^{-2x}}} = \dfrac{2}{\sqrt{e^{2x} - 4}}$

$\therefore f^\backslash(x) = g^\backslash(x)$

$f(x) = g(x) + C$

$\sec^{-1}\left(\frac{e^x}{2}\right) = \sin^{-1}(-2e^{-x}) + C$

$\sec^{-1}\left(\frac{e^{\ln 2}}{2}\right) = \sin^{-1}(-2e^{-\ln 2}) + C$

at $x = \ln 2$

$\sec^{-1}(1) = \sin^{-1}(-1), + C$

$0 = -\frac{\pi}{2} + C \quad \rightarrow \quad C = \frac{\pi}{2}$

$\therefore \sec^{-1}\left(\frac{e^x}{2}\right) = \sin^{-1}(-2e^{-x}) + \frac{\pi}{2},$

Example 10

Verify the following identities

7 July 1997

$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \tan^{-1}(x)$ for, $x \geq 0$

Solution

$f(x) = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$$f^\backslash(x) = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} \cdot \frac{\sqrt{1+x^2} - x\left(\frac{2x}{2\sqrt{1+x^2}}\right)}{1+x^2} = \frac{1}{\sqrt{\frac{\left(\sqrt{1+x^2}\right)^2 - x^2}{\left(\sqrt{1+x^2}\right)^2}}} \cdot \frac{(1+x^2) - x^2}{\sqrt{1+x^2}(1+x^2)}$$

$$= \frac{\sqrt{1+x^2}}{\sqrt{\left(\sqrt{1+x^2}\right)^2 - x^2}} \cdot \frac{1}{\sqrt{1+x^2}(1+x^2)} = \frac{1}{\sqrt{1+x^2 - x^2}} \cdot \frac{1}{(1+x^2)} = \frac{1}{1+x^2}$$

$g(x) = \tan^{-1}(x)$ $\qquad g^\backslash(x) = \dfrac{1}{1+x^2}$

$\therefore f^\backslash(x) = g^\backslash(x)$

$f(x) = g(x) + C$

$\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \tan^{-1}(x) + C$

at $x = 0$

$\sin^{-1}(0) = \tan^{-1}(0) + C \quad \rightarrow \quad 0 = 0 + C \quad \rightarrow \quad C = 0$

$\therefore \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \tan^{-1}(x)$

Example 11 — 12 July 2000 A

Show that: $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$, for all $x > 0$

Solution

$f(x) = \tan^{-1}\left(\frac{1}{x}\right)$ $\qquad f^{\backslash}(x) = \dfrac{\frac{-1}{x^2}}{\left(1+\left(\frac{1}{x}\right)^2\right)} = -\dfrac{1}{x^2+1}$

$g(x) = \cot^{-1} x$ $\qquad g^{\backslash}(x) = -\dfrac{1}{x^2+1}$

$\therefore f^{\backslash}(x) = g^{\backslash}(x)$

$f(x) = g(x) + C$

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x + C$

at $x = 1$

$\tan^{-1}(1) = \cot^{-1}(1) + C \quad \rightarrow \quad \frac{\pi}{4} = \frac{\pi}{4} + C \quad \rightarrow \quad C = 0$

$$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$$

Example 12 Find y' if $y = \sqrt{\dfrac{(\ln x)(\tan^{-1} x)}{2^x\sqrt{1+3x}}}$ — 20 November 2006 A

Solution

$y = \frac{1}{2}\ln(\ln x) + \frac{1}{2}\ln(\tan^{-1} x) - \frac{1}{2}x\ln 2 - \frac{1}{4}\ln(1+3x)$

$y^{\backslash} = \dfrac{1}{2x\ln x} + \dfrac{1}{2\tan^{-1} x}\cdot\dfrac{1}{1+x^2} - \dfrac{1}{2}\ln 2 - \dfrac{3}{4(1+3x)}$

Example 13 Find $\dfrac{dy}{dx}$ if $y = \left(\dfrac{x^2\sin^{-1} x}{(1-2x)e^{2x}}\right)^3$ — 10 March 1999

Solution

$y = 6\ln x + 3\ln\sin^{-1} x - 3\ln(1-2x) - 6x$

$y^{\backslash} = \dfrac{6}{x} + \dfrac{3}{\sin^{-1} x}\cdot\dfrac{1}{\sqrt{1-x^2}} - \dfrac{3(-2)}{1-2x} - 6$

Example 14 Find $\dfrac{dy}{dx}$ if $y = (\cos x)^{(\cos x)} + \cos^{-1}(\log_3 x)$ — 6 March 1997

Solution

لان اللوغاريتم لا يتوزع على الجمع
فيجب تقسيم الدالة إلى دالتين ونشتق
كل دالة على حدة ثم يكون المشتقة
المطلوبة عبارة عن مجموع
مشتقتي هاتان الدالتان

$y_1 = (\cos x)^{(\cos x)} \qquad \ln y_1 = \cos x \ln(\cos x)$

$\dfrac{y^{\backslash}_1}{y_1} = -\sin x\ln(\cos x) + \cos x\dfrac{-\sin x}{\cos x}$

$y^{\backslash}_1 = [= -\sin x\ln(\cos x) - \sin x](\cos x)^{(\cos x)}$

$y_2 = \cos^{-1}(\log_3 x) \quad \rightarrow y^{\backslash}_2 = \dfrac{-1}{\sqrt{1-(\log_3 x)^2}}\cdot\dfrac{1}{x}\cdot\dfrac{1}{\ln 3}$

$y^{\backslash} = y^{\backslash}_1 + y^{\backslash}_2$

$$y^{\backslash} = [= -\sin x\ln(\cos x) - \sin x](\cos x)^{(\cos x)} + \dfrac{-1}{\sqrt{1-(\log_3 x)^2}}\cdot\dfrac{1}{x}\cdot\dfrac{1}{\ln 3}$$

Example 15 Use logarithmic differentiation to find y' if $y=\dfrac{\left(\dfrac{1}{x-1}\right)^{x}\tan^{-1}x}{e^{3x}\sqrt{x^2-1}}$

27 November 2008 A

Solution

$$\ln y=-x\ln(x-1)+\ln\tan^{-1}x-3x-\frac{1}{2}\ln(x^2-1)$$

$$\frac{y^\backslash}{y}=-\ln(x-1)-\frac{x}{x-1}+\frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2}-3-\frac{2x}{2(x^2-1)}$$

Example 16 Find $\dfrac{dy}{dx}$ if $y=\pi^{\cot x}\left(x^{\tan^{-1}x}\right)$ 23 Nov. 2007 A

Solution

$$y^\backslash=\cot x\ln\pi+\tan^{-1}x\ln x$$

$$\frac{y^\backslash}{y}=-\csc^2x\ln\pi+\frac{1}{1+x^2}\ln x+\frac{1}{x}\tan^{-1}x$$

$$y^\backslash=\pi^{\cot x}\left(x^{\tan^{-1}x}\right)\left[\frac{\ln x}{1+x^2}+\frac{\tan^{-1}x}{x}\csc^2x\ln\pi\right]$$

Example 17 Find $\dfrac{dy}{dx}$ if $\sin^{-1}(xy)-\tan^{-1}\left(\dfrac{x}{y}\right)=0$ 4 July 1996

Solution

$$\frac{1}{\sqrt{1+x^2y^2}}(y+xy^\backslash)+\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{y-xy^\backslash}{y^2}=0$$

$$y^\backslash\left[\frac{x}{\sqrt{1+x^2y^2}}-\frac{x}{1+\left(\frac{x}{y}\right)^2}\right]+\frac{y}{\sqrt{1+x^2y^2}}+\frac{y}{\left(1+\left(\frac{x}{y}\right)^2\right)y^2}=0$$

$$y^\backslash=\left[-\frac{y}{\sqrt{1+x^2y^2}}-\frac{1}{y\left(1+\left(\frac{x}{y}\right)^2\right)}\right]\left[\frac{x}{\sqrt{1+x^2y^2}}-\frac{x}{1+\left(\frac{x}{y}\right)^2}\right]^{-1}$$

Example 18 Find $\dfrac{dy}{dx}$ if $\tan^{-1}\dfrac{y}{x}=\pi^{\tan^{-1}x}+\sin^{-1}(3x-2y)$.

16 November 2004

Solution

$$\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{xy^\backslash-y}{x^2}=\pi^{\tan^{-1}x}\cdot\frac{1}{1+x^2}\cdot\ln\pi+\frac{3-2y^\backslash}{\sqrt{1-(3x-2y)^2}}$$

$$y^\backslash\left[\frac{x}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}+\frac{2}{\sqrt{1-(3x-2y)^2}}\right]=\frac{\pi^{\tan^{-1}x}}{1+x^2}\cdot\ln\pi+\frac{y}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}+\frac{3}{\sqrt{1-(3x-2y)^2}}$$

$$y^\backslash=\left[\frac{\pi^{\tan^{-1}x}}{1+x^2}\cdot\ln\pi+\frac{y}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}+\frac{3}{\sqrt{1-(3x-2y)^2}}\right]\left[\frac{x}{x^2+y^2}+\frac{2}{\sqrt{1-(3x-2y)^2}}\right]^{-1}$$

Homework

1	Find the derivative of $y = \tan^{-1}(xe^{-x}) + \ln[(3^x + 1)(2 + \cos x)]$	31 August 2008 A
2	Find $\frac{dy}{dx}$ if $y = \ln\sqrt{\frac{\sin(e^x)}{\sqrt{x} + \sec^{-1} x}}$	16 November 2004
3	Find the derivative of $y = \ln[\log_{10}(\sin^{-1} x)] + x^{\cos^{-1} x}$	2 May 1995
4	Use logarithmic differentiation to find $f^\backslash(0)$ when [3 mark] $f(x) = \frac{(x+1)^\pi (2\arcsin x - 3)e^x}{\sqrt{x^2+1}}$	31 10 July 2010

Homework

1	Prove the following identity: $\cos(2\sin^{-1} x) = 1 - 2x^2$	24 March 2008 A
2	Prove the identity: $2\sin^{-1}(\sqrt{x}) = \cos^{-1}(1 - 2x)$, $(0 \le x \le 1)$	27 Nov. 2008 A
3	Show that $\cot^{-1}\left(\frac{\sqrt{1-x}}{\sqrt{x}}\right) = \sin^{-1}(\sqrt{x})$	18 July 2005 A
4	Show that $\cos^{-1}\left(\frac{x-1}{x+1}\right) = 2\cot^{-1}(\sqrt{x})$	22 July 2007
5	Show that: $\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)$, for all $x > 0$	
6	Show that: $\tan(\sin^{-1} x) - \cot(\cos^{-1} x) = 0$, for $x \in (0, 1)$	13 March 2001 A

7	Show that: $\sin(\tan^{-1} x - \cos^{-1} x) = \dfrac{x^2 - \sqrt{1-x^2}}{\sqrt{x^2+1}}$	1 May 1994
8	Prove the following identity: $\cos^{-1}(-x) = \pi - \cos^{-1} x$	1 January 1995
9	(3 pts) Show that $\sin(2\tan^{-1} x) = \dfrac{2x}{1+x^2}$ for all $x \in \mathbb{R}$.	30 April 11, 2010
10	(2 pts) Show that $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{1+x^2}}$ for any $x \in (-\infty, \infty)$	33 April 10, 2011
11	39 June 4, 2011 A Determine whether the statement is true or false . Justify your answer for credit . (1 pt each) (b) $\csc\left(\sin^{-1}\left(\frac{1}{x}\right)\right) = x$ for all $\lvert x \rvert \geq 1$.	
12	Use implicit differentiation Find $\dfrac{dy}{dx}$ If $\tan^{-1}(y) = \ln\lvert 1+xy \rvert$	

12 Use implicit differentiation Find $\dfrac{dy}{dx}$ If $\tan^{-1}(y) = \ln\lvert 1+xy \rvert$

Solution

$$\frac{y'}{1+y^2} = \frac{1}{1+xy}(y + xy')$$

$$y'\left[\frac{1}{1+y} - \frac{x}{1+xy}\right] = \frac{y}{1+xy}$$

$$y' = \frac{y}{1+xy}\left[\frac{1}{1+y} - \frac{x}{1+xy}\right]^{-1}$$